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STRESS ANALYSIS OF A STRETCHED SLAB HAVING A SPHERICAL CAVITY UNDER UNIDIRECTIONAL TENSION

DOO-SUNG LEE
Department of Mathematics, Konkuk University, 93-1 Mojin-Dong, Sungdong-Gu, Seoul, Korea

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Abstract—This paper presents the asymmetric analysis of the stress distribution arising in an isotropic infinite slab with a symmetrically located spherical cavity under a unidirectional uniform tension. To satisfy both boundary conditions on the surface of the slab and the surface of the cavity, harmonic functions in rectangular and spherical coordinates are used and the double Fourier transform is applied. The problem is reduced to the solution of infinite sets of linear equations which are suitable for numerical solution by an iterative technique. Finally some numerical examples are given for various values of the radius of the cavity.

1. INTRODUCTION

The problem of stress concentration has been mostly studied for infinite media. Although there have been some investigations concerning a medium of finite thickness, these have been mostly restricted to the axisymmetric cases, and the asymmetric problems received little attention. In the present paper, the problem of determining the stress distribution in a slab of an isotropic material having a spherical cavity which is subjected to unidirectional uniform tension is considered.

The problem of stress distribution in a stretched slab of isotropic material having a spherical cavity subjected to all-round tension has been considered by Ling (1959). This problem was also considered by Fox (1960) and Kaufman (1958), and Ling (1969) has also investigated the problem involving an eccentric spherical cavity. On the other hand, Atsumi and Itou (1973) have considered the case of a transversely isotropic material. All of the above papers are concerned with axisymmetric analysis, and the problem when the uniform tension is unidirectional, which is more physically realistic appears not to have been investigated. The purpose of this paper is to analyse the stresses when the tension is unidirectional, instead of all-round. To solve the present problem, the required stress functions are constructed by combining harmonic functions of three independent variables in rectangular and spherical coordinates and then the Fourier transform is used.

The problem of the stress concentration in the vicinity of a spherical cavity has been considered by Southwell (1926). For the elastic inclusion of different materials, Goodier's work (1933) provided a detailed analysis of stress concentration. Subsequently, Sadowsky and Sternberg (1947) derived the stress field around an ellipsoidal cavity under the plane stress condition, and Edwards (1951) considered the specific case of a spheroidal inclusion and cavity. We cite some more recent works which are relevant to the present problem. Zureick (1989) and Zureick and Eubanks (1988) considered the problem of an infinite transversely isotropic medium containing a spheroidal cavity when either asymmetric tractions or displacements are prescribed on the surface of the cavity. Hadama and Kodama (1986) obtained a solution to the problem of an elastic infinite body containing a series of spherical cavities under tension. The axisymmetric problem regarding the stress distribution at a spherical cavity in a circular cylinder which was first considered by Ling (1956) was investigated for the general case of axisymmetric loading by Solyanik-Krassa (1986). Tandon and Weng (1986) considered the problem of stress concentration when the medium contains finite concentrations of inclusions, and the compressibility of an isolated spheroidal cavity in an isotropic medium was considered by Zimmerman (1985).

2. SOLUTION PROCEDURE

Denote as usual the cylindrical, and spherical coordinates of a point by (r, θ, z) and (ρ, ϕ, θ), respectively. Two coordinate systems are connected to each other by

$$z = \rho\cos\phi = \rho\mu, \quad r = \rho\sin\phi = \rho\sqrt{1-\mu^2}. \tag{1}$$

Consider an infinite slab of thickness $2h$ having a spherical cavity of radius λ located symmetrically between the surfaces of the slab shown in Fig. 1.

The asymmetric stress components in rectangular coordinates which satisfy the equilibrium equations are expressed in terms of potential functions F, Z and ψ in Green and Zerna (1968) as follows.

$$\sigma_{xz} = \frac{\partial^2 F}{\partial x\,\partial z} + \frac{\partial^2 \psi}{\partial y\,\partial z} + z\frac{\partial^2 Z}{\partial x\,\partial z} - (1-2v)\frac{\partial Z}{\partial x}, \tag{2a}$$

$$\sigma_{yz} = \frac{\partial^2 F}{\partial y\,\partial z} - \frac{\partial^2 \psi}{\partial x\,\partial z} + z\frac{\partial^2 Z}{\partial y\,\partial z} - (1-2v)\frac{\partial Z}{\partial y}, \tag{2b}$$

$$\sigma_{xy} = \frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} + z\frac{\partial^2 Z}{\partial x\,\partial y}, \tag{2c}$$

$$\sigma_{zz} = \frac{\partial^2 F}{\partial z^2} + z\frac{\partial^2 Z}{\partial z^2} - 2(1-v)\frac{\partial Z}{\partial z}, \tag{2d}$$

$$\sigma_{xx} = \frac{\partial^2 F}{\partial x^2} + 2\frac{\partial^2 \psi}{\partial x\,\partial y} + z\frac{\partial^2 Z}{\partial x^2} - 2v\frac{\partial Z}{\partial z}, \tag{2e}$$

$$\sigma_{yy} = \frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 \psi}{\partial x\,\partial y} + z\frac{\partial^2 Z}{\partial y^2} - 2v\frac{\partial Z}{\partial z}, \tag{2f}$$

where

$$\nabla^2 F = \nabla^2 \psi = \nabla^2 Z = 0$$

with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In eqn (2) v is Poisson's ratio. From eqn (2), we can immediately see that in the absence of the cavity, the basic stress functions

$$F_1 = T(1-v)(x^2 - z^2)/2(1+v), \tag{3a}$$

$$Z_1 = -Tz/2(1+v), \tag{3b}$$

$$\psi_1 = Tvxy/2(1+v), \tag{3c}$$

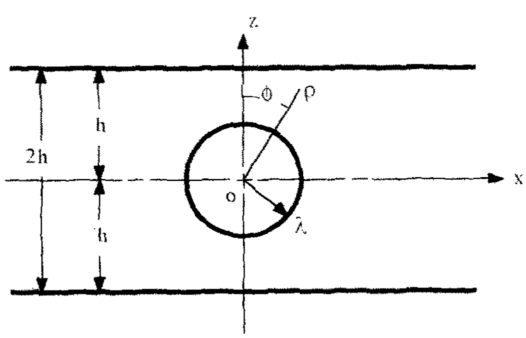

Fig. 1. Coordinate system.

will produce no stresses except the uniform tension T in the x-direction. In the sequel, T is deleted for convenience.

Suitable potential functions for the present problem which give rise to displacement and stress fields are

$$F(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} A(\xi,\eta)\cosh(\zeta z)\exp[-i(\xi x+\eta y)]\,d\xi\,d\eta + \sum_{n=0}^{\infty} \rho^{-2n-1}$$
$$\times \{a_n P_{2n}(\mu) + a_n^{(2)}\cos(2\theta)P_{2n}^2(\mu)\} - \alpha\frac{1}{\rho}[P_0^{-2}(\mu)+\tfrac{1}{2}Q_0^2(\mu)]\cos(2\theta), \quad \text{(4a)}$$

$$Z(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} B(\xi,\eta)\sinh(\zeta z)\exp[-i(\xi x+\eta y)]\,d\xi\,d\eta + \sum_{n=0}^{\infty} \rho^{-2n-2}$$
$$\times \{b_n P_{2n+1}(\mu) + b_n^{(2)}\cos(2\theta)P_{2n+1}^2(\mu)\} - \beta\frac{1}{\rho^2}[3P_1^{-2}(\mu)-\tfrac{1}{2}Q_1^2(\mu)]\cos(2\theta), \quad \text{(4b)}$$

$$\psi(x,y,z) = \left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} C(\xi,\eta)\cosh(\zeta z)\exp[-i(\xi x+\eta y)]\,d\xi\,d\eta$$
$$+\sum_{n=1}^{\infty} \rho^{-2n-1}c_n^{(2)}\sin(2\theta)P_{2n}^2(\mu) - \gamma\frac{1}{\rho}[P_0^{-2}(\mu)+\tfrac{1}{2}Q_0^2(\mu)]\sin(2\theta), \quad \text{(4c)}$$

where α, β and γ are the constants which will be determined subsequently. In the representation (4), the coefficients $a_0^{(2)}$, and $b_0^{(2)}$ are absent since $P_n^2(\mu) = 0$, when $2 > n$. Also, $\zeta = \sqrt{\xi^2+\eta^2}$ and P_n, P_n^2, P_n^{-2} and Q_n^2 are the Legendre and the associated Legendre functions of the first and second kind, respectively, and these functions are defined in Erdelyi *et al.* (1953) as follows:

$$P_n^{-2}(\mu) = (1-\mu^2)^{-1}\int_1^{\mu}\int_1^{\mu} P_n(\mu)(d\mu)^2,$$
$$Q_n^2(\mu) = (1-\mu^2)\frac{d^2Q_n(\mu)}{d\mu^2},$$

where the required expressions of the Legendre functions of the second kind in the present work are

$$Q_0(\mu) = \tfrac{1}{2}\log\frac{\mu+1}{\mu-1}, \quad Q_1(\mu) = \tfrac{1}{2}\mu\log\frac{\mu+1}{\mu-1} - 1.$$

The boundary conditions which are to be satisfied by the stress functions are

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \quad \text{on} \quad z = h \quad (5)$$

and

$$\sigma_{\rho\rho} = \sigma_{\rho\phi} = \sigma_{\rho\theta} = 0 \quad \text{on} \quad \rho = \lambda. \quad (6)$$

We shall determine $A(\xi,\eta)$, $B(\xi,\eta)$ and $C(\xi,\eta)$ from the boundary conditions (5) on the surface of the slab. The boundary condition $\sigma_{xz} = 0$ on $z = h$ gives, after Fourier transform, the following equation:

$$A(\xi,\eta)\xi\zeta\sinh(\zeta h)+C(\xi,\eta)\eta\zeta\sinh(\zeta h)+B(\xi,\eta)\{\zeta h\cosh(\zeta h)-(1-2\nu)\sinh(\zeta h)\}$$

$$= -\left[\xi\frac{\partial}{\partial z}\mathscr{F}\{F_0\}+\eta\frac{\partial}{\partial z}\mathscr{F}\{\psi_0\}+\xi\left[z\frac{\partial}{\partial z}-(1-2\nu)\right]\mathscr{F}\{Z_0\}\right]_{z=h}, \quad (7)$$

where F_0, Z_0 and ψ_0 are the third and fourth terms of eqns (4a), (4b) and (4c), respectively, and $\mathscr{F}\{\cdot\}$ is the Fourier transform defined by,

$$\mathscr{F}\{F_0\} = \int\int_{-\infty}^{\infty} F_0(\rho,\phi,\theta)\exp[i(\xi x+\eta y)]\,dx\,dy \quad (8)$$

and $\mathscr{F}\{\psi_0\}$ and $\mathscr{F}\{Z_0\}$ are defined in a similar way. To evaluate integrals in eqn (8), we continue as follows. If we take polar coordinates, defined by the equations

$$x = r\cos\theta,\quad y = r\sin\theta$$

we have

$$\int\int_{-\infty}^{\infty}\frac{P_n^m(\cos\phi)}{\rho^{n+1}}\exp[i(\xi x+\eta y)]\cos(m\theta)\,dx\,dy$$

$$= \int_0^{\infty}\frac{P_n^m(z(z^2+r^2)^{-1/2})}{(z^2+r^2)^{(n+1)/2}}\left(\int_0^{2\pi}\cos(m\theta)\exp[ir(\xi\cos\theta+\eta\sin\theta)]\,d\theta\right)r\,dr. \quad (9)$$

Now letting

$$\xi = \zeta\cos u,\quad \eta = \zeta\sin u, \quad (10)$$

the inner integral in the right-hand side of eqn (9) can be written as

$$\int_0^{2\pi}\cos(m\theta)\exp[ir\zeta\cos(\theta-u)]\,d\theta = 2\pi i^m\cos(mu)J_m(\zeta r) \quad (11)$$

by Whittaker and Watson (1948). If we now make use of the integral formula found in Erdelyi *et al.* (1954)

$$\int_0^{\infty}\frac{P_n^{-m}(z(z^2+r^2)^{-1/2})}{(z^2+r^2)^{(n+1)/2}}J_m(\zeta r)r\,dr = \frac{\zeta^{n-1}e^{-z\zeta}}{\Gamma(n+m+1)} \quad (12)$$

and the relation

$$P_n^m(\mu) = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}P_n^{-m}(\mu)$$

then, eqn (9) becomes,

$$\int\int_{-\infty}^{\infty}\frac{P_n^m(\cos\phi)}{\rho^{n+1}}\exp[i(\xi x+\eta y)]\cos(m\theta)\,dx\,dy = 2\pi i^m\cos(mu)\frac{\zeta^{n-1}e^{-z\zeta}}{(n-m)!}. \quad (13a)$$

Similarly

$$\iint_{-\infty}^{\infty} \frac{P_n^m(\cos\phi)}{\rho^{n+1}} \exp[\mathrm{i}(\xi x+\eta y)] \sin(m\theta)\,\mathrm{d}x\,\mathrm{d}y = 2\pi \mathrm{i}^m \sin(mu) \frac{\zeta^{n-1}\,\mathrm{e}^{-z\zeta}}{(n-m)!}. \tag{13b}$$

Now, using the relation

$$\frac{Q_0^2(\cos\phi)}{\rho} = \frac{2z}{r^2},$$

from the analyses similar to the previous procedures, we find that

$$\iint_{-\infty}^{\infty} \frac{Q_0^2(\cos\phi)}{\rho} \cos(2\theta) \exp[\mathrm{i}(\xi x+\eta y)]\,\mathrm{d}x\,\mathrm{d}y = -4\pi z \cos(2u) \int_0^{\infty} \frac{J_2(\zeta r)\,\mathrm{d}r}{r}$$
$$= -2\pi z \cos(2u). \tag{14}$$

In the last equation, we have made use of the relation in Erdelyi *et al.* (1954)

$$\int_0^{\infty} \frac{J_2(\zeta r)\,\mathrm{d}r}{r} = \frac{1}{2}. \tag{15}$$

Similarly, we have

$$\iint_{-\infty}^{\infty} \frac{Q_0^2(\cos\phi)}{\rho} \sin(2\theta) \exp[\mathrm{i}(\xi x+\eta y)]\,\mathrm{d}x\,\mathrm{d}y = -2\pi z \sin(2u). \tag{16}$$

Next, since

$$\frac{Q_1^2(\cos\phi)}{\rho^2} = -\frac{2}{r^2},$$

we have

$$\iint_{-\infty}^{\infty} \frac{Q_1^2(\cos\phi)}{\rho^2} \cos(2\theta) \exp[\mathrm{i}(\xi x+\eta y)]\,\mathrm{d}x\,\mathrm{d}y = 2\pi \cos(2u). \tag{17}$$

Thus, if we substitute eqns (13), (14), (16) and (17) into eqn (7), we obtain one relation connecting $A(\xi,\eta)$, $B(\xi,\eta)$ and $C(\xi,\eta)$:

$$A(\xi,\eta)\xi\zeta \sinh(\zeta h) + B(\xi,\eta)\xi\{\zeta h \cosh(\zeta h) - (1-2\nu)\sinh(\zeta h)\} + C(\xi,\eta)\eta\zeta \sinh(\zeta h)$$
$$= 2\pi\,\mathrm{e}^{-\zeta h}[\xi A' + \xi\{\zeta h + (1-2\nu)\}B' + \eta C'] - \pi[\{\alpha + (1-2\nu)\beta\}\xi \cos(2u) + \gamma\eta \sin(2u)], \tag{18}$$

where A', B' and C' are defined as

$$A' = \sum_{n=0}^{\infty} \frac{a_n \zeta^{2n}}{(2n)!} - \cos(2u)\left\{\sum_{n=1}^{\infty} \frac{a_n^{(2)}\zeta^{2n}}{(2n-2)!} + \frac{\alpha}{2}\right\}, \tag{19a}$$

$$B' = \sum_{n=0}^{\infty} \frac{b_n \zeta^{2n}}{(2n+1)!} - \cos(2u)\left\{\sum_{n=1}^{\infty} \frac{b_n^{(2)}\zeta^{2n}}{(2n-1)!} - \frac{\beta}{2}\right\}, \tag{19b}$$

$$C' = -\sin(2u)\left\{\sum_{n=1}^{\infty} \frac{c_n^{(2)}\zeta^{2n}}{(2n-2)!} + \frac{\gamma}{2}\right\}. \tag{19c}$$

Similarly, the remaining two boundary conditions $\sigma_{yz} = 0$, and $\sigma_{zz} = 0$ on $z = h$ lead to the following two additional equations for solving $A(\xi,\eta)$, $B(\xi,\eta)$ and $C(\xi,\eta)$:

$$A(\xi,\eta)\eta\zeta \sinh(\zeta h) + B(\xi,\eta)\eta\{\zeta h \cosh(\zeta h) - (1-2\nu)\sinh(\zeta h)\} - C(\xi,\eta)\xi\zeta \sinh(\zeta h)$$
$$= 2\pi e^{-\zeta h}[\eta A' + \eta\{\zeta h + 1 - 2\nu\}B' - \xi C'] - \pi[\{\alpha + (1-2\nu)\beta\}\eta \cos(2u) - \gamma\xi \sin(2u)], \quad (20)$$

$$A(\xi,\eta)\zeta \cosh(\zeta h) + B(\xi,\eta)\{\zeta h \sinh(\zeta h) - 2(1-\nu)\cosh(\zeta h)\}$$
$$= 2\pi e^{-\zeta h}[A' + \{\zeta h + 2(1-\nu)\}B']. \quad (21)$$

Therefore, we now have three equations, (18), (20) and (21), to solve for $A(\xi,\eta)$, $B(\xi,\eta)$ and $C(\xi,\eta)$ in terms of A', B' and C'. Thus, after solving these equations, we obtain:

$$A(\xi,\eta) = \frac{2\pi\zeta^{-1}}{2\zeta h + \sinh(2\zeta h)}(-A'\{2\zeta h - 3 + 4\nu - e^{-2\zeta h}\} - 2B'\{\zeta h - (1-2\nu)(2-2\nu)\}$$
$$+ \{\alpha + (1-2\nu)\beta\}[\zeta h \sinh(\zeta h) - 2(1-\nu)\cosh(\zeta h)]\cos(2u)), \quad (22a)$$

$$B(\xi,\eta) = \frac{2\pi}{2\zeta h + \sinh(2\zeta h)}(2A' + \{2\zeta h + 3 - 4\nu - e^{-2\zeta h}\}B'$$
$$- \{\alpha + (1-2\nu)\beta\}\cosh(\zeta h)\cos(2u)), \quad (22b)$$

$$C(\xi,\eta) = \frac{\pi\zeta^{-1}}{\sinh(\zeta h)}(2 e^{-\zeta h} C' + \gamma \sin(2u)). \quad (22c)$$

If we substitute the value of $C(\xi,\eta)$ from eqn (23c) into the second term of eqn (4c), we obtain

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} C(\xi,\eta)\cosh(\zeta z)\exp[-i(\xi x + \eta y)]\, d\xi\, d\eta = -\frac{1}{2\pi}\int\int_{-\infty}^{\infty} \frac{\sin(2u)}{\sinh(\zeta h)\zeta}$$
$$\times\left(e^{-\zeta h}\sum_{n=1}^{\infty}\frac{c_n^{(2)}}{(2n-2)!}\zeta^{2n} + \frac{\gamma}{2}(e^{-\zeta h} - 1)\right)\cosh(\zeta z)\exp[-i(\xi x + \eta y)]\, d\xi\, d\eta. \quad (23)$$

To satisfy boundary conditions on the surface of spherical cavity, it is necessary to express eqn (23) in spherical coordinates, and this is performed as follows. If we use eqn (10), the first integral on the right-hand side of eqn (23) can be written as

$$\int\int_{-\infty}^{\infty} \frac{\sin(2u)\zeta^{2n-1}}{\sinh(\zeta h)} e^{-\zeta h}\cosh(\zeta z)\exp[-i(\xi x + \eta y)]\, d\xi\, d\eta$$
$$= \int_0^{\infty}\frac{\zeta^{2n} e^{-\zeta h}}{\sinh(\zeta h)}\int_{-\pi}^{\pi}\cosh(\zeta z)\exp[-i\zeta(x\cos u + y \sin u)]\sin(2u)\, du\, d\zeta. \quad (24)$$

From Whittaker and Watson (1948), we can easily obtain the following formula:

$$\int_{-\pi}^{\pi}\cosh(\zeta z)\exp[-i\zeta(x\cos u + y\sin u)]\sin(2mu)\, du$$
$$= (-1)^m 2\pi \sin(2m\theta)\sum_{k=m}^{\infty}\frac{\zeta^{2k}}{(2k+2m)!}\rho^{2k}P_{2k}^{2m}(\cos\phi), \quad (m = 1). \quad (25)$$

Therefore, if we employ eqn (25) in eqn (23), it can be written as

$$\left(\frac{1}{2\pi}\right)^2 \int\int_{-\infty}^{\infty} C(\xi,\eta)\cosh(\zeta z)\exp[-i(\xi x+\eta y)]\,d\xi\,d\eta$$

$$= \sum_{k=1}^{\infty}\left[\sum_{n=1}^{\infty}\left[\frac{I_{n+k}c_n^{(2)}}{(2n-2)!}\right]+\frac{\gamma}{2}\int_0^{\infty}\frac{\zeta^{2k}(e^{-\zeta h}-1)}{\sinh(\zeta h)}\,d\zeta\right]\frac{\rho^{2k}}{(2k+2)!}P_{2k}^2(\cos\phi), \quad (26)$$

where

$$I_{n+k} = \int_0^{\infty}\frac{e^{-\zeta h}\,\zeta^{2n+2k}}{\sinh(\zeta h)}\,d\zeta. \quad (27)$$

Therefore the stress function $\psi(x, y, z)$ in eqn (4c) can be expressed in spherical coordinates as follows:

$$\psi(\rho,\phi,\theta) = \sin(2\theta)\sum_{k=1}^{\infty}\left[\frac{\Psi_k}{(2k+2)!}\rho^{2k}+c_k^{(2)}\rho^{-2k-1}\right]P_{2k}^2(\cos\phi), \quad (28)$$

where

$$\Psi_k = \sum_{n=1}^{\infty}\frac{I_{n+k}c_n^{(2)}}{(2n-2)!}+\Theta_k, \quad (29)$$

$$\Theta_k = \frac{\gamma}{2}\left(I_k-\int_0^{\infty}\frac{\zeta^{2k}\,d\zeta}{\sinh(\zeta h)}\right). \quad (30)$$

To express $F(x, y, z)$ in spherical coordinates, we need the following formula:

$$\int_{-\pi}^{\pi}\cosh(\zeta z)\exp[-i\zeta(x\cos u+y\sin u)]\cos(2mu)\,du$$

$$= (-1)^m 2\pi\cos(2m\theta)\sum_{k=m}^{\infty}\frac{\zeta^{2k}}{(2k+2m)!}\rho^{2k}P_{2k}^{2m}(\cos\phi), \quad (m = 0, 1). \quad (31)$$

Similarly, if we use eqn (31), we can express $F(x, y, z)$ in spherical coordinates as follows:

$$F(\rho,\phi,\theta) = \sum_{k=0}^{\infty}\left[\frac{\phi_k}{(2k)!}\rho^{2k}+a_k\rho^{-2k-1}\right]P_{2k}(\cos\phi)$$

$$+\cos(2\theta)\sum_{k=1}^{\infty}\left[\frac{\phi_k^{(2)}}{(2k+2)!}\rho^{2k}+a_k^{(2)}\rho^{-2k-1}\right]P_{2k}^2(\cos\phi), \quad (32)$$

where

$$\phi_k = -\sum_{n=0}^{\infty}\left[\frac{L_{n+k}a_n}{(2n)!}+\frac{M_{n+k}b_n}{(2n+1)!}\right], \quad (33)$$

$$\phi_k^{(2)} = -\sum_{n=1}^{\infty}\left[\frac{L_{n+k}a_n^{(2)}}{(2n-2)!}+\frac{M_{n+k}b_n^{(2)}}{(2n-1)!}\right]+\Pi_k, \quad (34)$$

with

$$L_{n+k} = \int_0^\infty \frac{\zeta^{2n+2k}(2\zeta h - 3 + 4\nu - e^{-2\zeta h})}{2\zeta h + \sinh(2\zeta h)}\, d\zeta, \tag{35}$$

$$M_{n+k} = \int_0^\infty \frac{2\zeta^{2n+2k}\{\zeta^2 h^2 - (1-2\nu)(2-2\nu)\}}{2\zeta h + \sinh(2\zeta h)}\, d\zeta, \tag{36}$$

$$\Pi_k = -\tfrac{1}{2}(\alpha L_k - \beta M_k) - \{\alpha + \beta(1-2\nu)\} \int_0^\infty \frac{\zeta h \sinh(\zeta h) - 2(1-\nu)\cosh(\zeta h)}{2\zeta h + \sinh(2\zeta h)} \zeta^{2k}\, d\zeta. \tag{37}$$

Equations (35) and (36) contain terms which are divergent at the lower limit if $(n+k) = 0$. However, the divergence can be removed by adding $(1-\nu)/\zeta h$ and $(1-2\nu)(2-2\nu)/2\zeta h$ to eqns (35) and (36), respectively. These divergent terms are so-called trivial terms and do not affect the stresses within the slab.

Now we use the following formula to express $Z(x, y, z)$ in spherical coordinates:

$$\int_{-\pi}^{\pi} \sinh(\zeta z) \exp[-i\zeta(x\cos u + y \sin u)] \cos(2mu)\, du$$

$$= (-1)^m 2\pi \cos(2m\theta) \sum_{k=m}^{\infty} \frac{(\zeta\rho)^{2k+1}}{(2k+1+2m)!} P_{2k+1}^{2m}(\cos\phi), \quad (m = 0, 1). \tag{38}$$

Thus eqn (4b) becomes

$$Z(\rho, \phi, \theta) = \sum_{k=0}^{\infty} \left[\frac{\chi_k}{(2k+1)!}\rho^{2k+1} + b_k \rho^{-2k-2}\right] P_{2k+1}(\cos\phi)$$

$$+\cos(2\theta) \sum_{k=1}^{\infty} \left[\frac{\chi_k^{(2)}}{(2k+3)!}\rho^{2k+1} + b_k^{(2)}\rho^{-2k-2}\right] P_{2k+1}^2(\cos\phi), \tag{39}$$

where

$$\chi_k = \sum_{n=0}^{\infty} \left[\frac{J_{n+k} a_n}{(2n)!} + \frac{K_{n+k} b_n}{(2n+1)!}\right], \tag{40}$$

$$\chi_k^{(2)} = \sum_{n=1}^{\infty} \left[\frac{J_{n+k} a_n^{(2)}}{(2n-2)!} + \frac{K_{n+k} b_n^{(2)}}{(2n-1)!}\right] + \Gamma_k, \tag{41}$$

with

$$J_{n+k} = \int_0^\infty \frac{2\zeta^{2n+2k+2}}{2\zeta h + \sinh(2\zeta h)}\, d\zeta, \tag{42}$$

$$K_{n+k} = \int_0^\infty \frac{(2\zeta h + 3 - 4\nu - e^{-2\zeta h})}{2\zeta h + \sinh(2\zeta h)} \zeta^{2n+2k+2}\, d\zeta, \tag{43}$$

$$\Gamma_k = \tfrac{1}{2}(\alpha J_k - \beta K_k) + \{\alpha + \beta(1-2\nu)\} \int_0^\infty \frac{\cosh(\zeta h)}{2\zeta h + \sinh(2\zeta h)} \zeta^{2k+2}\, d\zeta. \tag{44}$$

To satisfy the boundary conditions on the surface of the cavity, we need stress components in spherical coordinates in terms of F, ψ and Z. These are expressible as follows:

$$\sigma_{\rho\rho} = \frac{\partial^2 F}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial^2 \psi}{\partial \rho\, \partial \theta} - \frac{2}{\rho^2}\frac{\partial \psi}{\partial \theta} + \rho\frac{\partial^2 Z}{\partial \rho^2}\cos\phi - 2(1-\nu)\frac{\partial Z}{\partial \rho}\cos\phi + 2\nu\frac{\partial Z}{\partial \phi}\frac{\sin\phi}{\rho}, \tag{45}$$

$$\sigma_{\rho\phi} = \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial F}{\partial \phi}\right) + \cot\phi\, \rho\frac{\partial}{\partial \rho}\left(\frac{1}{\rho^2}\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi\, \partial \theta} + \frac{\partial}{\partial \phi}\left(\frac{\partial Z}{\partial \rho}\cos\phi\right)$$
$$+2(1-\nu)\left[\frac{\partial Z}{\partial \rho}\sin\phi - \frac{\partial Z}{\partial \rho}\frac{\cos\phi}{\rho}\right], \tag{46}$$

$$\sigma_{\rho\theta} = \frac{1}{\sin\phi}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial F}{\partial \theta}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \rho}\right)\rho\sin\phi - \rho\cos\phi\frac{\partial}{\partial \rho}\left(\frac{1}{\rho^2}\frac{\partial \psi}{\partial \phi}\right)$$
$$+\cot\phi\frac{\partial^2 Z}{\partial \rho\, \partial \theta} - 2(1-\nu)\frac{\partial Z}{\partial \theta}\frac{\cot\phi}{\rho}, \tag{47}$$

$$\sigma_{\phi\phi} = \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{2}{\rho^2}\frac{\partial}{\partial \phi}\left(\cot\phi\frac{\partial \psi}{\partial \theta}\right) + \frac{2}{\rho^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{\rho}\frac{\partial^2 Z}{\partial \phi^2}\cos\phi$$
$$+2(1-\nu)\frac{\sin\phi}{\rho}\frac{\partial Z}{\partial \phi} + (1-2\nu)\cos\phi\frac{\partial Z}{\partial \rho}. \tag{48}$$

The stress components due to the uniform tension are denoted by superscript (1), and when expressed in spherical coordinates, using eqns (45)–(48), are as follows:

$$\sigma^{(1)}_{\rho\rho} = \tfrac{1}{6}P^2_2(\cos\phi)\cos(2\theta) - \tfrac{1}{3}P_2(\cos\phi) + \tfrac{1}{3}P_0(\cos\phi)$$
$$= \tfrac{1}{2}\sin^2\phi(\cos(2\theta)+1), \tag{49a}$$

$$\sigma^{(1)}_{\rho\phi} = \frac{1}{30\sin\phi}P^2_3(\cos\phi)\cos(2\theta) + \tfrac{1}{6}\sin\phi\, \mathrm{d}P_2(\mu)/\mathrm{d}\mu$$
$$= \tfrac{1}{2}\sin\phi\cos\phi\,(\cos(2\theta)+1), \tag{49b}$$

$$\sigma^{(1)}_{\rho\theta} = -\frac{1}{6\sin\phi}P^2_2(\cos\phi)\sin(2\theta) = -\tfrac{1}{2}\sin\phi\sin(2\theta), \tag{49c}$$

$$\sigma^{(1)}_{\phi\phi} = \tfrac{1}{2}\cos^2\phi(\cos(2\theta)+1). \tag{49d}$$

Now, the stress components $\sigma_{\rho\rho}$, $\sigma_{\rho\phi}$ and $\sigma_{\rho\theta}$ are all zero on the surface of the spherical cavity, $\rho = \lambda$. The condition $\sigma_{\rho\rho} = 0$ on $\rho = \lambda$ gives one equation to determine a_n and b_n, i.e.

$$\sum_{k=0}^{\infty}\left\{\frac{\phi_k}{(2k)!}2k(2k-1)\lambda^{2k-2} + a_k(2k+1)(2k+2)\lambda^{-2k-3}\right.$$
$$+\left[\frac{\chi_{k-1}}{(2k-1)!}(2k-1)(2k-4+4\nu)\lambda^{2k-2} + b_{k-1}(2k(2k+3)-2\nu)\lambda^{-2k-1}\right]\frac{2k}{4k-1}$$
$$\left.+\left[\frac{\chi_k}{(2k+1)!}\{2k(2k-1)-2-2\nu\}\lambda^{2k} + b_k(2k+2)(2k+5-4\nu)\lambda^{-2k-3}\right]\frac{2k+1}{4k+3}\right\}$$
$$\cdot P_{2k}(\cos\phi) + \tfrac{1}{3}P_0(\cos\phi) - \tfrac{1}{3}P_2(\cos\phi) = 0. \tag{50}$$

Setting the coefficients of $P_{2k}(\cos\phi)$ equal to zero, we obtain one equation for determining a_n and b_n:

$$\sum_{n=0}^{\infty} \{a_n G_1(n,k) + b_n G_2(n,k)\} + a_k z_1(k) + b_{k-1} z_2(k) + b_k z_3(k) = \tfrac{1}{3}\delta_{k,1} - \tfrac{1}{3}\delta_{k,0}, \quad k \geqslant 0, \quad (51)$$

where δ is Kronecker's delta and

$$G_1(n,k) = (p_1 L_{n+k} + p_2 J_{n+k-1} + p_3 J_{n+k}) t_1, \quad (52)$$

$$G_2(n,k) = (p_1 M_{n+k} + p_2 K_{n+k-1} + p_3 K_{n+k}) t_1/(2n+1), \quad (53)$$

with

$$\begin{aligned}
p_1 &= -2k(2k-1)/\lambda^2, \\
p_2 &= (2k)^2(2k-1)(2k-4+4v)/(4k-1)\lambda^2, \\
p_3 &= \{2k(2k-1)-2-2v\}/(4k+3), \\
t_1 &= \lambda^{2k}/(2n)!(2k)!, \\
z_1(k) &= (2k+1)(2k+2)/\lambda^{2k+3}, \\
z_2(k) &= \{2k(2k+3)-2v\}2k/(4k-1)\lambda^{2k+1}, \\
z_3(k) &= (2k+2)(2k+1)(2k+5-4v)/(4k+3)\lambda^{2k+3}.
\end{aligned}$$

Similarly, the boundary condition $\sigma_{\rho\phi} = 0$ on $\rho = \lambda$ gives another equation to determine a_n and b_n, i.e.

$$\sum_{k=0}^{\infty} \Bigg\{ -\frac{\phi_k}{(2k)!}(2k-1)\lambda^{2k-2} + a_k(2k+2)\lambda^{-2k-3} - \Bigg[\frac{\chi_{k-1}}{(2k-1)!}(2k-1)(2k-4+4v)\lambda^{2k-2}$$

$$+ b_{k-1}\{(2k-1)(2k-5)+1-2v\}\lambda^{-2k-1}\Bigg]\Bigg/(4k-1) - \Bigg[\frac{\chi_k}{(2k+1)!}\{2k(2k+2)-1+2v\}\lambda^{2k}$$

$$+ b_k(2k+2)(2k+5-4v)\lambda^{-2k-3}\Bigg]\Bigg/(4k+3)\Bigg\} P'_{2k}(\mu) + \tfrac{1}{6}P'_2(\mu) = 0, \quad (54)$$

where prime indicates the differentiation with respect to the argument. If we set the coefficients of $P'_{2k}(\mu)$ equal to zero, we obtain,

$$\sum_{n=0}^{\infty} \{a_n H_1(n,k) + b_n H_2(n,k)\} + a_k \frac{z_1(k)}{(2k+1)} + b_{k-1} w_2(k) + b_k \frac{z_3(k)}{(2k+1)} = -\tfrac{1}{6}\delta_{k,1}, \quad k \geqslant 1, \quad (55)$$

where

$$H_1(n,k) = \left(-\frac{p_1}{2k} L_{n+k} - \frac{p_1}{2k} J_{n+k-1} + q_3 J_{n+k}\right) t_1, \quad (56)$$

$$H_2(n,k) = \left(-\frac{p_1}{2k} M_{n+k} - \frac{p_1}{2k} K_{n+k-1} + q_3 K_{n+k}\right) t_1/(2n+1), \quad (57)$$

with

$$\begin{aligned}
q_3 &= -\{2k(2k+2)-1+2v\}/(2k+1)(4k+3), \\
w_2(k) &= -\{(2k-1)(2k-5)+1-2v\}/(4k-1)\lambda^{2k+1}.
\end{aligned}$$

Equations (51) and (55) are best solved for a_n and b_n by iteration.

In an analogous way, from three boundary conditions $\sigma_{\rho\rho} = \sigma_{\rho\phi} = \sigma_{\rho\theta} = 0$ on $\rho = \lambda$, we can obtain three equations for determining $a_n^{(2)}$, $b_n^{(2)}$ and $c_n^{(2)}$. For this purpose, we notice that the functions appearing in the third terms of eqn (4) take the forms:

$$P_0^{-2}(\mu) + \tfrac{1}{2}Q_0^2(\mu) = \frac{1+\cos^2\phi}{2\sin^2\phi}, \tag{58}$$

$$3P_1^{-2}(\mu) - \tfrac{1}{2}Q_1^2(\mu) = \frac{\cos^3\phi - 3\cos\phi}{2\sin^2\phi}. \tag{59}$$

We also need the following two relations found in Erdelyi *et al.* (1953) concerning the associated Legendre functions:

$$(2n+1)P_n^m(\cos\phi)\cos\phi = (n-m+1)P_{n+1}^m(\cos\phi) + (n+m)P_{n-1}^m(\cos\phi), \tag{60}$$

$$\frac{dP_n^m(\cos\phi)}{d\phi} = n\cot\phi P_n^m(\cos\phi) - \frac{n+m}{\sin\phi}P_{n-1}^m(\cos\phi). \tag{61}$$

Thus we have

$$\frac{dP_n^2(\cos\phi)}{d\phi}\sin\phi = \frac{1}{2n+1}[n(n-1)P_{n+1}^2(\cos\phi) - (n+2)(n+1)P_{n-1}^2(\cos\phi)]. \tag{62}$$

Altogether, eqns (4), (45), (49), (58), (59) and (62), and the boundary condition $\sigma_{\rho\rho} = 0$ on $\rho = \lambda$ yield the following equation:

$$\begin{aligned}\sum_{k=1}^{\infty} \Bigg\{ & [-\phi_k^{(2)}p_{11} + \Psi_k r_{11} + \chi_{k-1}^{(2)}q_{11} + \chi_k^{(2)}q_{12}]\frac{\lambda^{2k}}{(2k+2)!} \\ & + a_k^{(2)}f_{11}(k) + b_{k-1}^{(2)}g_{11}(k) + b_k^{(2)}g_{12}(k) + c_k^{(2)}h_{11}(k)\Bigg\} P_{2k}^2(\cos\phi)\cos(2\theta) \\ & + \frac{1}{\lambda^3}\cos(2\theta)\left[\left(1 - \frac{2}{\sin^2\phi}\right)\{\alpha - 4\gamma + (5-4v)\beta\} + \beta(5-v)\sin^2\phi\right] \\ & + \tfrac{1}{2}\sin^2\phi\cos(2\theta) = 0,\end{aligned} \tag{63}$$

where the constants $p_{11}, r_{11}, \ldots,$ are given in the Appendix. The function $1/\sin^2\phi$ is singular when $\phi = 0$. To get a regular solution over this region, we set its coefficient equal to zero. Thus we have one relation connecting α, β and γ:

$$\alpha + (5-4v)\beta - 4\gamma = 0. \tag{64}$$

Next we need the following two relations:

$$\begin{aligned}\frac{dP_n^2(\cos\phi)}{d\phi}\cos\phi = \frac{1}{\sin\phi(2n+1)}\Bigg\{ & \frac{n^2(n-1)}{2n+3}P_{n+2}^2(\cos\phi) - \left[\frac{n(n-1)(n+3)}{2n+3}\right. \\ & \left. + \frac{(n+1)(n+2)(n-2)}{2n-1}\right]P_n^2(\cos\phi) - \frac{(n+2)(n+1)^2}{2n-1}P_{n-2}^2(\cos\phi)\Bigg\},\end{aligned} \tag{65}$$

$$P_n^2(\cos\phi)\sin\phi = \frac{1}{\sin\phi(2n+1)}\left\{-\frac{n(n-1)}{2n+3}P_{n+2}^2(\cos\phi)+\left[2n+1\right.\right.$$

$$\left.\left.-\frac{(n-1)(n+3)}{2n+3}-\frac{(n+2)(n-2)}{2n-1}\right]P_n^2(\cos\phi)-\frac{(n+2)(n+1)}{2n-1}P_{n-2}^2(\cos\phi)\right\} \quad (66)$$

to use the boundary condition $\sigma_{\rho\phi} = 0$ on $\rho = \lambda$ to obtain another relation to determine the unknown constants. Thus we get

$$\sum_{k=2}^{\infty}\left\{[\phi_{k-1}^{(2)}p_{21}+\phi_k^{(2)}p_{22}+\Psi_{k-1}r_{21}+\Psi_k r_{22}+\chi_{k-2}^{(2)}q_{21}+\chi_{k-1}^{(2)}q_{22}\right.$$

$$+\chi_k^{(2)}q_{23}]\frac{\lambda^{2k-2}}{(2k)!}+a_{k-1}^{(2)}f_{21}(k)+a_k^{(2)}f_{22}(k)+b_{k-2}^{(2)}g_{21}(k)+b_{k-1}^{(2)}g_{22}(k)$$

$$\left.+b_k^{(2)}g_{23}(k)+c_{k-1}^{(2)}h_{21}(k)+c_k^{(2)}h_{22}(k)\right\}P_{2k-1}^2(\cos\phi)\cos(2\theta)/\sin\phi$$

$$+\frac{1}{\lambda^3}\cos(2\theta)\left[-\frac{2\cos\phi}{\sin^2\phi}\{2\alpha-5\gamma+(4-2\nu)\beta\}+\frac{\cos\phi}{\sin\phi}\{-3\gamma+6(1-\nu)\beta\}\right.$$

$$\left.-\beta(1+\nu)\cos\phi\sin\phi\right]+\tfrac{1}{2}\sin\phi\cos\phi\cos(2\theta) = 0, \quad (67)$$

where the constants $p_{21},\ldots,$ are listed in the Appendix. Again, to remove the singularity in eqn (67), we set

$$2\alpha-5\gamma+(4-2\nu)\beta = 0. \quad (68)$$

Since the coefficient of $\cos\phi/\sin\phi$ in eqn (67) is the linear combination of eqns (64) and (68), the singular terms in eqn (67) are removed.

The condition $\sigma_{\rho\theta} = 0$ on $\rho = \lambda$, after using eqns (65) and (66) yields the following equation:

$$\sum_{k=1}^{\infty}\left\{[\phi_k^{(2)}p_{31}+\Psi_k r_{31}+\Psi_{k+1}r_{32}+\chi_{k-1}^{(2)}q_{31}+\chi_k^{(2)}q_{32}]\frac{\lambda^{2k}}{(2k+2)!}+a_k^{(2)}f_{31}(k)\right.$$

$$\left.+b_{k-1}^{(2)}g_{31}(k)+b_k^{(2)}g_{32}(k)+c_{k-1}^{(2)}h_{31}(k)+c_k^{(2)}h_{32}(k)\right\}\frac{P_{2k}^2(\cos\phi)}{\sin\phi}\sin(2\theta)$$

$$+\frac{\sin(2\theta)}{\lambda^3}\left[\{2\alpha-5\gamma+(4-2\nu)\beta\}\left(\frac{1}{\sin\phi}-\frac{2}{\sin^3\phi}\right)+\sin\phi\{-\tfrac{3}{2}\gamma+(4-2\nu)\beta\}\right]$$

$$-\tfrac{1}{2}\sin\phi\sin(2\theta) = 0. \quad (69)$$

The singular terms in the last equation vanish since their coefficient is identically zero from eqn (68). Now solving eqns (64) and (68) for α and γ in terms of β, we find that

$$\alpha = (3-4\nu)\beta, \quad \gamma = 2(1-\nu)\beta. \quad (70)$$

In eqns (63), (67) and (69), as the thickness h tends to infinity, the summed terms vanish. The coefficients of each Legendre function vanish. If we examine these terms, we see that these terms contain $a_k^{(2)}(k \geqslant 2)$, $b_k^{(2)}(k \geqslant 1)$, and $c_k^{(2)}(k \geqslant 1)$. Thus the only remaining coefficient is $a_1^{(2)}$. Therefore, we have from eqns (63) and (67), when $h = \infty$

$$3a_1^{(2)}\frac{12}{\lambda^5}+\frac{\beta(5-\nu)}{\lambda^3}=-\tfrac{1}{2}, \tag{71a}$$

$$-3a_1^{(2)}\frac{8}{\lambda^5}-\frac{\beta(1+\nu)}{\lambda^3}=-\tfrac{1}{2}, \tag{71b}$$

and from eqn (69) we will again obtain eqn (71b). Solving eqns (71a) and (71b) simultaneously for $a_1^{(2)}$ and β, we find that

$$\beta=\frac{5\lambda^3}{10\nu-14},\quad a_1^{(2)}=-\frac{\lambda^5}{2(10\nu-14)}. \tag{72}$$

Therefore setting $\beta=5\lambda^3/(10\nu-14)$, from eqns (63), (67) and (69), we obtain the following three equations for determining unknown coefficients $a_k^{(2)}$, $b_k^{(2)}$ and $c_k^{(2)}$. From eqn (63), we have

$$\sum_{n=1}^{\infty}\{a_n^{(2)}\alpha_1(n,k)+b_n^{(2)}\beta_1(n,k)+c_n^{(2)}\gamma_1(n,k)\}+a_k^{(2)}f_{11}(k)+b_{k-1}^{(2)}g_{11}(k)$$
$$+b_k^{(2)}g_{12}(k)+c_k^{(2)}h_{11}(k)+[-\Pi_k p_{11}+\Theta_k r_{11}$$
$$+\Gamma_{k-1}q_{11}+\Gamma_k q_{12}]\frac{\lambda^{2k}}{(2k+2)!}=-\frac{5\nu-1}{10\nu-14}\delta_{k,1},\quad k\geqslant 1, \tag{73}$$

and from eqn (67) we get

$$\sum_{n=1}^{\infty}\{a_n^{(2)}\alpha_2(n,k)+b_n^{(2)}\beta_2(n,k)+c_n^{(2)}\gamma_2(n,k)\}+a_{k-1}^{(2)}f_{21}(k)+a_k^{(2)}f_{22}(k)$$
$$+b_{k-2}^{(2)}g_{21}(k)+b_{k-1}^{(2)}g_{22}(k)+b_k^{(2)}g_{23}(k)+c_{k-1}^{(2)}h_{21}(k)+c_k^{(2)}h_{22}(k)+[\Pi_{k-1}p_{21}+\Pi_k p_{22}$$
$$+\Theta_{k-1}r_{21}+\Theta_k r_{22}+\Gamma_{k-2}q_{21}+\Gamma_{k-1}q_{22}+\Gamma_k q_{23}]\frac{\lambda^{2k-2}}{(2k)!}=\frac{12}{15(10\nu-14)}\delta_{k,2},\quad k\geqslant 2, \tag{74}$$

and eqn (69) yields

$$\sum_{n=1}^{\infty}\{a_n^{(2)}\alpha_3(n,k)+b_n^{(2)}\beta_3(n,k)+c_n^{(2)}\gamma_3(n,k)\}+a_k^{(2)}f_{31}(k)+b_{k-1}^{(2)}g_{31}(k)$$
$$+b_k^{(2)}g_{32}(k)+c_{k-1}^{(2)}h_{31}(k)+c_k^{(2)}h_{32}(k)+[\Pi_k p_{31}+\Theta_k r_{31}+\Theta_{k+1}r_{32}$$
$$+\Gamma_{k-1}q_{31}+\Gamma_k q_{32}]\frac{\lambda^{2k}}{(2k+2)!}=-\frac{4}{10\nu-14}\delta_{k,1},\quad \mathrm{k}\geqslant 1. \tag{75}$$

The detailed expressions of α_k, β_k and γ_k ($k=1,2,3$) are given in the Appendix.

3. NUMERICAL EXAMPLE

Numerical examples will be given for the various values of λ. We can calculate the stresses at any point on the slab. However, the most important one is the maximum stress occurring at the poles of the cavity, i.e. at $(\rho,\mu)=(\lambda,\pm 1)$. From eqns (48) and (49d), and the value $P_k^{(2)}(0)=3[(5)_{k-2}/(k-2)!]$, we find that

$$\sigma_{\phi\phi} = -\frac{1}{\lambda^2}\sum_{k=0}^{\infty}\left[\frac{\phi_k}{(2k)!}k(2k-1)\lambda^{2k}+a_k(k+1)(2k+1)\lambda^{-2k-1}\right]+\frac{1}{\lambda^2}\sum_{k=1}^{\infty}\left[\frac{\phi_k^{(2)}}{(2k+2)!}\lambda^{2k}\right.$$
$$\left.+a_k^{(2)}\lambda^{-2k-1}\right]6\frac{(5)_{2k-2}}{(2k-2)!}\cos(2\theta)-\frac{1}{\lambda}\sum_{k=0}^{\infty}\left[\frac{\chi_k}{(2k+1)!}(k+2\nu)(2k+1)\lambda^{2k-1}\right.$$
$$\left.+b_k(k+1)(2k+3-4\nu)\lambda^{-2k-3}\right]+\frac{1}{\lambda}\sum_{k=1}^{\infty}\left[\frac{\chi_k^{(2)}}{(2k+3)!}\lambda^{2k+1}+b_k^{(2)}\lambda^{-2k-2}\right]$$
$$\times 6\frac{(5)_{2k-1}}{(2k-1)!}\cos(2\theta)+\frac{2}{\lambda^2}\sum_{k=1}^{\infty}\left[\frac{\Psi_k}{(2k+2)!}\lambda^{2k}+c_k^{(2)}\lambda^{-2k-1}\right]6\frac{(5)_{2k-2}}{(2k-2)!}\cos(2\theta)$$
$$+\tfrac{1}{2}\{\cos(2\theta)+1\}+\frac{\cos(2\theta)}{\lambda^3}\left[-\frac{\alpha}{2}+2\gamma+\beta(4\nu-\tfrac{7}{2})\right]. \tag{76}$$

Of interest is the limiting value of $\sigma_{\phi\phi}$ as the thickness of the plate h tends to infinity. In this case we have seen earlier that all the coefficients $a_k^{(2)}$, $b_k^{(2)}$ and $c_k^{(2)}$ vanish except $a_1^{(2)}$ which is given by the second of eqns (72). To find limiting values of a_n and b_n, we look at eqns (50) and (54). As $h \to \infty$, all ϕ_k and χ_k tend to zero, and coefficients of each Legendre function in summed terms in eqns (50) and (54) vanish if $k \geqslant 2$, and these terms contain $a_k (k \geqslant 2)$, and $b_k (k \geqslant 1)$. Therefore, the only remaining coefficients are a_0, a_1 and b_0. Thus, from eqn (50) we obtain

$$2a_0+\frac{10-8\nu}{3}b_0 = -\tfrac{1}{3}\lambda^3. \tag{77}$$

$$\frac{12}{\lambda^2}a_1+\frac{4(5-\nu)}{3}b_0 = \tfrac{1}{3}\lambda^3. \tag{78}$$

On the other hand, from eqn (54) we obtain

$$\frac{4}{\lambda^2}a_1+\frac{2(1+\nu)}{3}b_0 = -\tfrac{1}{6}\lambda^3. \tag{79}$$

If we solve eqns (78) and (79) simultaneously for a_1 and b_0, we find that

$$a_1 = -\frac{\lambda^5}{14-10\nu}, \quad b_0 = \frac{5\lambda^3}{2(14-10\nu)} \tag{80}$$

and from these values and eqn (77), we obtain

$$a_0 = -\frac{(13-10\nu)}{2(14-10\nu)}\lambda^3. \tag{81}$$

In eqn (76), as $h \to \infty$, all ϕ_k, $\phi_k^{(2)}$, λ_k, $\chi_k^{(2)}$ and Ψ_k vanish. Thus eqn (76) reduces to

$$\sigma_{\phi\phi}^{(\infty)} = -6a_1\lambda^{-5}-a_0\lambda^{-3}-(3-4\nu)b_0\lambda^{-3}+a_1^{(2)}6\lambda^{-5}\cos(2\theta)+\tfrac{1}{2}\{\cos(2\theta)+1\}$$
$$+\frac{5(2\nu-1)}{10\nu-14}\cos(2\theta). \tag{82}$$

Putting the values of $a_1^{(2)}$, a_0, a_1 and b_0 from eqns (72), (81) and (80) into eqn (82), we find that

$$\sigma_{\phi\phi}^{(\infty)} = \frac{12}{14-10\nu} + \frac{15(1-\nu)}{14-10\nu}\cos(2\theta).$$

When $\theta = 0$ the above reduces to $(27-15\nu)/(14-10\nu)$ which is in complete agreement with the known result which can be found in Timoshenko and Goodier (1970).

Now in eqn (76), if we substitute ϕ_k in eqn (33) and so on, one of the double sums can be expressed in closed form, and $\sigma_{\phi\phi}$ can be written as follows:

$$\begin{aligned}
\sigma_{\phi\phi} = {} & \tfrac{1}{2} + \sum_{k=0}^{\infty} a_k\left(\frac{1}{(2k)!}\{\tfrac{1}{2}R_k - 4\nu S'_k - S_k\} - \lambda^{-2k-3}(k+1)(2k+1)\right) \\
& + \sum_{k=0}^{\infty} b_k\left(\frac{1}{(2k+1)!}\{U_k - 2\nu V'_k - \tfrac{1}{2}V_k\} - \lambda^{-2k-3}(k+1)(2k+3-4\nu)\right) \\
& + \cos(2\theta)\left(\sum_{k=1}^{\infty} a_k^{(2)}\frac{1}{(2k-2)!}\{R_k + 2S_k + \lambda^{-2k-3}6(5)_{2k-2}\} + \sum_{k=1}^{\infty} b_k^{(2)}\frac{1}{(2k-1)!}\right. \\
& \times\{2U_k + V_k + \lambda^{-2k-3}6(5)_{2k-1}\} + \sum_{k=1}^{\infty} c_k^{(2)}\frac{1}{(2k-2)!}\{2W_k + \lambda^{-2k-3}12(5)_{2k-2}\} \\
& \left. + \frac{15\nu-12}{10\nu-14}\right) + \frac{5\lambda^3}{2(10\nu-14)}\cos(2\theta)\left(-(2-3\nu)\int_0^{\infty} f_0(\lambda,\zeta)\{\zeta h\sinh(\zeta h)\right. \\
& - 2(1-\nu)\cosh(\zeta h)\}\,\mathrm{d}\zeta + (2-3\nu)\int_0^{\infty} g_0(\lambda,\zeta)\cosh(\zeta h)\,\mathrm{d}\zeta - (1-\nu)\int_0^{\infty} h_0(\lambda,\zeta)\,\mathrm{d}\zeta \\
& \left. + \frac{3-4\nu}{4}(-R_0 + 2S_0) + \tfrac{1}{4}(2U_0 - V_0) + (1-\nu)W_0\right),
\end{aligned} \tag{83}$$

where

$$R_k = \int_0^{\infty} f_k(\lambda,\zeta)(2\zeta h - 3 + 4\nu - \mathrm{e}^{-2\zeta h})\,\mathrm{d}\zeta,$$

$$S'_k = \int_0^{\infty} f_k(\lambda,\zeta)\,\mathrm{d}\zeta,$$

$$S_k = \int_0^{\infty} g_k(\lambda,\zeta)\,\mathrm{d}\zeta,$$

$$U_k = \int_0^{\infty} f_k(\lambda,\zeta)\{\zeta^2h^2 - (1-2\nu)(2-2\nu)\}\,\mathrm{d}\zeta,$$

$$V'_k = \int_0^{\infty} f_k(\lambda,\zeta)(2\zeta h + 3 - 4\nu - \mathrm{e}^{-2\zeta h})\,\mathrm{d}\zeta,$$

$$V_k = \int_0^{\infty} g_k(\lambda,\zeta)(2\zeta h + 3 - 4\nu - \mathrm{e}^{-2\zeta h})\,\mathrm{d}\zeta,$$

$$W_k = \int_0^{\infty} h_k(\lambda,\zeta)\,\mathrm{e}^{-\zeta h}\,\mathrm{d}\zeta,$$

with

$$f_k(\lambda,\zeta) = \frac{\zeta^{2k+2}\cosh(\lambda\zeta)}{2\zeta h+\sinh(2\zeta h)},$$

$$g_k(\lambda,\zeta) = \frac{\zeta^{2k+3}\lambda\sinh(\lambda\zeta)}{2\zeta h+\sinh(2\zeta h)},$$

$$h_k(\lambda,\zeta) = \frac{\zeta^{2k+2}\cosh(\lambda\zeta)}{\sinh(\zeta h)}.$$

Table 1(a). Coefficients a_n, b_n, $a_n^{(2)}$, $b_n^{(2)}$ and $c_n^{(2)}$ for $\lambda = 0.1$

n	a_n	b_n	n	$a_n^{(2)}$	$b_n^{(2)}$	$c_n^{(2)}$
0	−0.4563E−03	0.2173E−03	1	0.4344E−06	−0.1951E−11	0.7104E−09
1	−0.8694E−06	−0.9755E−10	2	−0.9850E−13	−0.1174E−15	0.7423E−13
2	0.4646E−13	0.6592E−14	3	−0.1119E−16	−0.5140E−20	0.5171E−17
3	0.3527E−17	0.1258E−18	4	−0.1032E−20	−0.5870E−25	0.3429E−21
4	0.1600E−21	0.3310E−23	5	−0.7518E−25	0.1328E−28	0.1890E−25
5	0.5996E−26	0.1135E−27	6	−0.4110E−29	0.1275E−32	0.8057E−30
6	0.2518E−30	0.4242E−32	7	−0.1697E−33	0.6519E−37	0.2663E−34
7	0.1075E−34	0.0000E+00	8	−0.5430E−38	0.2304E−41	0.6988E−39

Table 1(b). Coefficients a_n, b_n, $a_n^{(2)}$, $b_n^{(2)}$ and $c_n^{(2)}$ for $\lambda = 0.2$

n	a_n	b_n	n	$a_n^{(2)}$	$b_n^{(2)}$	$c_n^{(2)}$
0	−0.3641E−02	0.1731E−02	1	0.1381E−04	−0.4768E−08	0.1665E−06
1	−0.2777E−04	−0.9877E−07	2	−0.4088E−09	−0.2369E−11	0.2883E−09
2	0.1904E−09	0.1070E−09	3	−0.7436E−12	−0.9662E−15	0.3214E−12
3	0.2280E−12	0.3273E−13	4	−0.1095E−14	0.2523E−18	0.3427E−15
4	0.1658E−15	0.1380E−16	5	−0.1273E−17	0.1262E−20	0.3045E−18
5	0.9955E−19	0.7575E−20	6	−0.1112E−20	0.1571E−23	0.2090E−21
6	0.6692E−22	0.4531E−23	7	−0.7333E−24	0.1206E−26	0.1111E−24
7	0.4575E−25	0.0000E+00	8	−0.3751E−27	0.6629E−30	0.4680E−28

Table 1(c). Coefficients a_n, b_n, $a_n^{(2)}$, $b_n^{(2)}$ and $c_n^{(2)}$ for $\lambda = 0.3$

n	a_n	b_n	n	$a_n^{(2)}$	$b_n^{(2)}$	$c_n^{(2)}$
0	−0.1223E−01	0.5799E−02	1	0.1033E−03	−0.3066E−06	0.3616E−05
1	−0.2095E−03	−0.5612E−05	2	−0.5408E−07	−0.7124E−09	0.3389E−07
2	0.2554E−07	0.3068E−07	3	−0.4993E−09	−0.1322E−11	0.1920E−09
3	0.1451E−09	0.4764E−10	4	−0.3710E−11	0.2581E−14	0.1047E−11
4	0.5390E−12	0.1018E−12	5	−0.2177E−13	0.5043E−16	0.4775E−14
5	0.1640E−14	0.2828E−15	6	−0.9595E−16	0.3084E−18	0.1678E−16
6	0.5580E−17	0.8575E−18	7	−0.3196E−18	0.1186E−20	0.4556E−19
7	0.1933E−19	0.0000E+00	8	−0.8263E−21	0.3285E−23	0.9779E−22

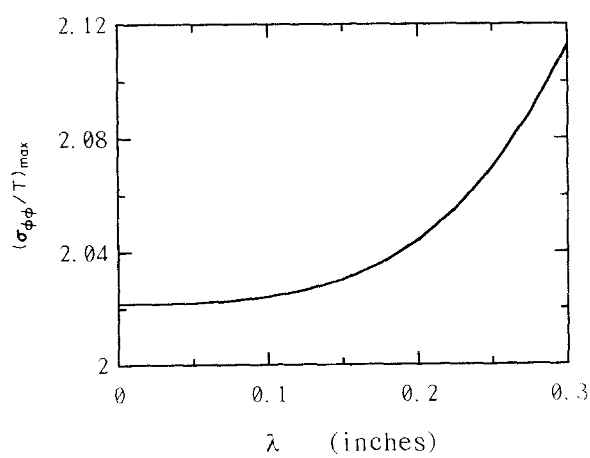

Fig. 2. Maximum stress versus radius of cavity.

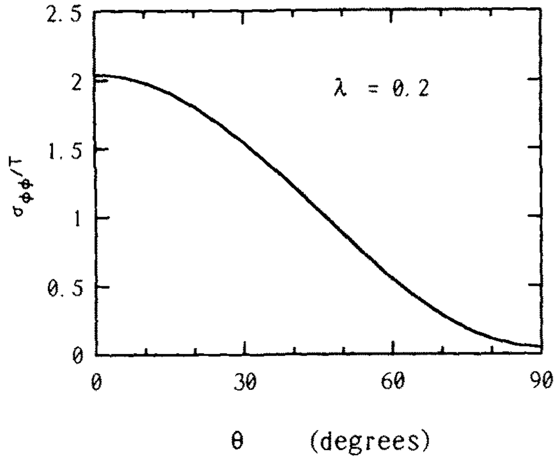

Fig. 3. Stress distribution around a spherical cavity $\lambda = 0.2''$.

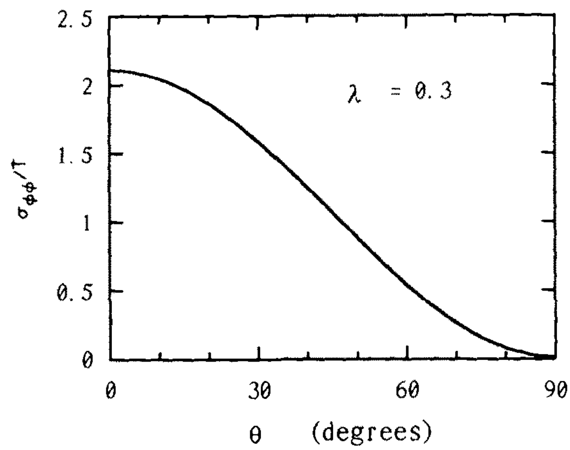

Fig. 4. Stress distribution around a spherical cavity $\lambda = 0.3''$.

Table 1 shows the values of the coefficients a_n, b_n, $a_n^{(2)}$, $b_n^{(2)}$ and $c_n^{(2)}$ corresponding to $\nu = 0.25$ and $h = 1''$. Using eqn (83), we evaluated the maximum stress at $(\rho, \mu) = (\lambda, 1)$ for various values of λ, and these are shown graphically in Fig. 2. Figures 3 and 4 show the stress distribution around the cavity for various values of θ when $\lambda = 0.2''$ and $\lambda = 0.3''$.

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APPENDIX

The detailed expressions of α, β, γ, f, g and h are as follows:

$$\alpha_1(n,k) = [-L_{n+k}p_{11} + J_{n+k-1}q_{11} + J_{n+k}q_{12}]s_1,$$
$$\beta_1(n,k) = [-M_{n+k}p_{11} + K_{n+k-1}q_{11} + K_{n+k}q_{12}]s_1/(2n-1),$$
$$\gamma_1(n,k) = r_{11}I_{n+k}s_1,$$

where

$$p_{11} = -2k(2k-1)/\lambda^2,$$
$$q_{11} = (2k-2)(2k-1)(2k+2)(2k-4+4\nu)/(4k-1)\lambda^2,$$
$$q_{12} = \{(2k+1)(2k-2)-2\nu\}/(4k+3),$$
$$r_{11} = 4(2k-1)/\lambda^2,$$
$$s_1 = \lambda^{2k}/(2n-2)!(2k+2)!,$$
$$f_{11}(k) = (2k+1)(2k+2)/\lambda^{2k+3},$$
$$g_{11}(k) = \{2k(2k+3)-2\nu\}(2k-2)/(4k-1)\lambda^{2k+1},$$
$$g_{12}(k) = (2k+2)(2k+3)(2k+5-4\nu)/(4k+3)\lambda^{2k+3},$$
$$h_{11}(k) = -8(k+1)/\lambda^{2k+3},$$

$$\alpha_2(n,k) = [-L_{n+k-1}p_{21} - L_{n+k}p_{22} + J_{n+k-2}q_{21} + J_{n+k-1}q_{22} + J_{n+k}q_{23}](2n-1)s_2,$$
$$\beta_2(n,k) = [-M_{n+k-1}p_{21} - M_{n+k}p_{22} + K_{n+k-2}q_{21} + K_{n+k-1}q_{22} + K_{n+k}q_{23}]s_2,$$
$$\gamma_2(n,k) = (I_{n+k-1}r_{21} + I_{n+k}r_{22})s_2,$$

where

$$p_{21} = (2k-3)^2(2k-2)/(4k-3)\lambda^2,$$
$$p_{22} = -(2k-1)/(4k+1),$$
$$q_{21} = (2k-4)(2k-3)^2(2k-6+4\nu)2k/(4k-5)(4k-3)\lambda^2,$$
$$q_{22} = [(1-2\nu)(2k-1) - (2k-2)^2(2k-1)(2k+2)/(4k-1)(4k+1)$$
$$\quad - (2k-3)(2k+1)\{2k(2k-2)-1+2\nu\}/(4k-3)(4k-1)]/(2k+1),$$
$$q_{23} = -\{2k+1+2\nu+(2k-1)(2k+2)\}\lambda^2/(4k+1)(4k+3),$$
$$r_{21} = 4(2k-3)^2/\lambda^2,$$
$$r_{22} = -4/(2k+1)(4k+1),$$
$$s_2 = \lambda^{2k-2}/(2n-1)!(2k)!,$$
$$f_{21}(k) = -2k(2k-2)(2k-3)/(4k-3)\lambda^{2k+1},$$
$$f_{22}(k) = (2k+2)^2(2k+1)/(4k+1)\lambda^{2k+3},$$
$$g_{21}(k) = -(2k-4)(2k-3)(2k(2k-4)+2+2\nu)/(4k-5)(4k-3)\lambda^{2k-1},$$

$$g_{22}(k) = [-2k(1-2\nu)+(2k-2)(2k+2)\left\{2k(1-2\nu)+\frac{(2k+2-2\nu)(2k-1)}{(4k-1)(4k+1)}\right\}$$

$$+(2k+1)2k(2k-3)(2k+3-4\nu)/(4k-1)(4k-3)]/\lambda^{2k+1},$$

$$g_{23}(k) = (2k+2)^2(2k+3)(2k+5-4\nu)/(4k+1)(4k+3)\lambda^{2k+3},$$

$$h_{21}(k) = -6(2k-3)/(4k-3)\lambda^{2k+1},$$

$$h_{22}(k) = -4(2k+2)^2/(4k+1)\lambda^{2k+1},$$

$$\alpha_3(n,k) = (-L_{n+k}p_{31}+J_{n+k-1}q_{31}+J_{n+k}q_{32})s_1$$

$$\beta_3(n,k) = (-M_{n+k}p_{31}+K_{n+k-1}q_{31}+K_{n+k}q_{32})s_1/(2n-1),$$

$$\gamma_3(n,k) = (I_{n+k}r_{31}+I_{n+k+1}r_{32})s_1,$$

where

$$p_{31} = -2(2k-1)/\lambda^2,$$

$$q_{31} = -2(2k-2)(2k+2)(2k-3+2\nu)/(4k-1)\lambda^2,$$

$$q_{32} = -2(2k-1+2\nu)/(4k+3),$$

$$r_{31} = \{-4+2(2k-2)[-k+(2k-1)(2k+3)2k/(4k+1)(4k+3)+2(k-1)(k+1)/(4k-1)]\}/\lambda^2,$$

$$r_{32} = 2k/(4k+3),$$

$$f_{31}(k) = 4(k+1)/\lambda^{2k+3},$$

$$g_{31}(k) = 8(k+1-\nu)(k-1)/(4k-1)\lambda^{2k+1},$$

$$g_{32}(k) = 4(k+2-\nu)(2k+3)/(4k+3)\lambda^{2k+3},$$

$$h_{31}(k) = (2k+1)(2k-3)(2k-2)/(4k-1)\lambda^{2k+1},$$

$$h_{32}(k) = [-4+(2k+3)\{-2k-1+(2k-1)(2k+3)/(4k+1)(4k+3)\}]/\lambda^{2k+3}.$$